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**Dynamical Superalgebra and Supersymmetry  
for a Many-Fermion System**

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DYNAMICAL SUPERALGEBRA AND SUPERSYMMETRY  
FOR A MANY-FERMION SYSTEM

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Abstract

A linearization scheme is proposed for the general Hamiltonian of an interacting fermion system, consisting of a mean field approximation in which pairing and umklapp play dominant roles. A variety of models emerge, characterized by a hierarchy of spectrum generating algebras and superalgebras, of which some are supersymmetric. The superconducting phase transition is shown to be related to supersymmetry breaking.

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It has been recently emphasized that interacting many-fermion systems can be handled by a method which is similar to that introduced by Solomon [1]. The strategy adopted in this method amounts essentially to linearizing the Hamiltonian by a generalized Hartree-Fock approximation leading to a conventional pair-reduced mean-field model. This linearization results in an effective Hamiltonian which is an element of a Lie algebra  $\mathcal{A}$ , referred to as spectrum generating algebra; the rotation which effects the diagonalization of the Hamiltonian, leading to the energy spectrum, is an automorphism of  $\mathcal{A}$ . The use of dynamical algebras permits a unified treatment of systems capable of simultaneously exhibiting more than one condensed phase [2].

In this paper we show how, starting from a general many-fermion interacting Hamiltonian, the reduction process can be extended to include more terms than in the standard case, and there results a hierarchy of spectrum generating algebras and superalgebras. The appearance of the latter is important not only for the diagonalization of the Hamiltonian; they also allow us to write the Hamiltonian in supersymmetric form [3].

The general Hamiltonian  $H$  for a system of fermions interacting via a two-body potential  $V$  is given by

$$H = \sum_i \epsilon_i a_i^\dagger a_i + 1/2 \sum_{i,j,l,k} \langle i j | V | k l \rangle a_i^\dagger a_j^\dagger a_l a_k \quad (1)$$

The index  $i$  on the annihilation operator  $a_i$  for a fermion includes both momentum and spin indices  $i = (\underline{i}, \sigma)$ , and the operators satisfy  $\{a_i, a_j^\dagger\} = \delta_{ij}$ ,  $\{a_i, a_j\} = 0$ .

Methods of solution usually involve reducing the four-fermion term in the interaction to quadratic terms, a process which we refer to as linearization. The resulting approximate (mean-field) Hamiltonian, being quadratic in  $a_i$ ,  $a_j^\dagger$  is then diagonalizable. One method of linearization is the following: we consider the interaction term in (1) as being the product of two operators  $A$  and  $B$ , and write the identity

$$AB = (A - \langle A \rangle)(B - \langle B \rangle) + \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle \quad (2)$$

We may interpret  $\langle A \rangle$  and  $\langle B \rangle$  as being the respective expectations of the operators A and B in some state. If we only consider states in our model for which the first term on the r.h.s. of (3) is "small" in some sense, we may approximate

$$AB \sim \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle \quad (3)$$

However, this approximation is consistent only in the following circumstances:

- (i) A, B commute; then BA leads to the same linear approximation, with  $\langle A \rangle, \langle B \rangle$  ordinary c-numbers;
- (ii) A, B anticommute; then  $BA = -AB$  implies that  $\langle A \rangle, \langle B \rangle$  must be thought of as anticommuting variables, which anticommute with the operators A, B as well.

However in case (ii) it is in fact sufficient, for our purposes here, to demand that these variables anticommute with the operators, since the term  $\langle A \rangle \langle B \rangle$  plays no role in the calculations, giving simply an additive constant in the Hamiltonian. For instance, if we consider the average values  $\langle a_{k_1} \dots a_{k_m} \rangle$  of a product of an odd number m of fermion operators  $a_k$ , this condition is implemented by imposing

$$\langle a_{k_1} \dots a_{k_m} \rangle = \eta \alpha_{k_1, \dots, k_m} \quad (4)$$

where  $\alpha_{k_1, \dots, k_m}$  is a c-number, and  $\eta$  a variable satisfying

$$\{\eta, a_k\} = 0 \quad \text{for all } k, \quad \eta^2 = - \text{Identity} \quad (5)$$

We note from (5) that  $\eta$  is not a Grassmann variable.

In the conventional linearization referred to above, for example in the BCS model of superconductivity, where A and B are commuting fermion pairs, then case (i) is operative. We shall also consider the case where A and B are single or triple fermion operators, and then we shall have case (ii).

In case (i) reduction of the Hamiltonian (1) leads to a bilinear form which is an element of a Lie algebra, the spectrum generating algebra of the model. In the BCS case, for example, on which the additional constraint of pairing in equal and opposite momentum and spin states is imposed, this leads essentially to  $su(2)$  [4]. In case (ii), we are lead naturally to a superalgebra.

We first consider a model for an interacting electron gas in a crystal lattice, and retain only the following terms in the interaction of (1), which can be treated in the framework of case (i):

- a)  $1/2 \sum_{i,j} \langle i -i | V | j -j \rangle a_i^\dagger a_{-i}^\dagger a_j a_{-j}$ ; the Cooper-pairing terms, responsible for superconductivity, and
- b)  $1/2 \sum'_{i,j} \langle i j | V | -j -i \rangle a_i^\dagger a_j^\dagger a_{-i} a_{-j}$ ; which refer to an umklapp process [5], permitted if momentum is conserved modulo a vector of the reciprocal lattice  $L$  of the crystal; thus  $2(\underline{i} + \underline{j}) \in L$  (the prime on the above summation indicates this limitation).

With this approximation, the Hamiltonian (1) reduces to  $H^{(u)}$ , a direct sum of commuting single- $k$  Hamiltonians,  $H^{(u)} = \sum_{\mathbf{k}} H_{\mathbf{k}}^{(u)}$ ;

$$H_{\mathbf{k}}^{(u)} = \epsilon_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}) + (\Delta_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + u_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}} + \text{h.c.}) \quad (6)$$

after using the linearization technique of case (i), taking  $A = B$  as  $a_i^\dagger a_{-i}^\dagger$  and  $a_i^\dagger a_i$ , successively (we have fixed  $\mathbf{k} = (\underline{k}, \uparrow)$ ,  $-\mathbf{k} = (-\underline{k}, \downarrow)$ ).

The  $\Delta_{\mathbf{k}}$  term is the conventional superconductivity one  $\Delta_{\mathbf{k}} = 1/2 \sum_j \langle \mathbf{k} -\mathbf{k} | V | \mathbf{j} -\mathbf{j} \rangle \langle a_j a_{-j} \rangle$ , while  $u_{\mathbf{k}}$  arises from the type of umklapp process considered above,  $u_{\mathbf{k}} = 1/2 \sum_j \langle \mathbf{k} \mathbf{j} | V | -\mathbf{j} -\mathbf{k} \rangle \langle a_j^\dagger a_{-j} \rangle$ . If we disregard the umklapp term ( $u_{\mathbf{k}}=0$ ) then we recover the BCS reduced model, for which the spectrum generating algebra is  $su(2)$ , generated by

$$J_+ = J_-^\dagger = a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger, \quad J_3 = 1/2 (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} - 1) \quad (7)$$

The additional operators in (8) generate a commuting  $su(2)$ ,

$$J'_+ = J'^{\dagger}_- = a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}, \quad J'_3 = 1/2 (a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}) \quad (8)$$

The spectrum generating algebra of this model is therefore  $su(2) \oplus su(2)$ , and the spectrum is immediately obtainable since

$$H_{\kappa}^{(1)} \mapsto \sqrt{\varepsilon_{\kappa}^2 + |\Delta_{\kappa}|^2} (a_{\kappa}^{\dagger} a_{\kappa} + a_{-\kappa}^{\dagger} a_{-\kappa} - 1) + |u_{\kappa}| (a_{\kappa}^{\dagger} a_{\kappa} - a_{-\kappa}^{\dagger} a_{-\kappa}) \quad (9)$$

under a rotation in  $su(2) \oplus su(2)$  space. The interest in  $H_{\kappa}^{(1)}$  as given by eq. (6) is that it may be regarded as the Hamiltonian of a model exhibiting spontaneously broken supersymmetry.

The principal model we wish to consider in this note, whose dynamical algebra will be a superalgebra, includes additional terms coming from the following umklapp processes:

$$c) \quad 1/2 \sum_{i, \kappa} \langle i - i | V | k i \rangle a_i^{\dagger} a_{-i}^{\dagger} a_i a_{\kappa} \quad ; \quad (\underline{i} + \underline{k} \in L)$$

$$d) \quad 1/2 \sum_{i, \kappa} \langle i - i | V | k - i \rangle a_i^{\dagger} a_{-i}^{\dagger} a_i a_{\kappa} \quad ; \quad (\underline{i} - \underline{k} \in L)$$

and their hermitian conjugates. We linearize these terms in the framework of case (ii), so that, for example,  $a_i^{\dagger} a_{-i}^{\dagger} a_i a_{\kappa} \sim \langle a_i^{\dagger} a_{-i}^{\dagger} a_i \rangle a_{\kappa} + a_i^{\dagger} a_{-i}^{\dagger} a_i \langle a_{\kappa} \rangle$ , where, following the prescription discussed before, we have dropped the constant  $\langle a_i^{\dagger} a_{-i}^{\dagger} a_i \rangle \langle a_{\kappa} \rangle$ , and the expectations  $\langle \rangle$  are given by the expressions (4).

By this method we obtain a new reduced Hamiltonian,  $H^{(2)} = \sum_{\kappa} H_{\kappa}^{(2)}$ , where the operators in  $H_{\kappa}^{(2)}$  are, in addition to those in  $H_{\kappa}^{(1)}$ , (with  $a_{+} \equiv a_{\kappa}$ ,  $a_{-} \equiv a_{-\kappa}$ ,  $n_{\pm} \equiv a_{\pm}^{\dagger} a_{\pm}$ ),

$$\{a_{+}(1-n_{-}), a_{-}(1-n_{+}), a_{+}^{\dagger}(1-n_{-}), a_{-}^{\dagger}(1-n_{+}), n_{+}a_{-}, n_{-}a_{+}, a_{-}^{\dagger}n_{+}, a_{+}^{\dagger}n_{-}\} = \{g_i | i=1, \dots, 8\} \quad (10)$$

Together with the six operators of (7), (8),

$$\{1/2 (n_{+} + n_{-} - 1), a_{+}^{\dagger} a_{-}^{\dagger}, a_{-} a_{+}, 1/2 (n_{+} - n_{-}), a_{+}^{\dagger} a_{-}, a_{-}^{\dagger} a_{+}\}, \quad (11)$$

these eight operators generate the 14-dimensional superalgebra  $A(1,1) = su(2|2)/\mathbb{C}I$  [6], [7]. The reduced Hamiltonian  $H_{\kappa}^{(2)}$  is a linear combination of elements of a representation of the compact real form of this superalgebra, with coefficients in the extended field  $\mathbb{R}[\eta]$  ( $\mathbb{R}[\eta] = \{a + \eta b | a, b \in \mathbb{R}\}$ ). The coefficients of the bosonic terms (11) are in



the even part of  $\mathbb{R}[\eta]$  (isomorphic with  $\mathbb{R}$ ), while those of the fermionic terms (10) are in the odd part ( $\eta\mathbb{R}$ ):

$$H_{\kappa}^{(2)} = H_{\kappa}^{(1)} + \eta \sum_{g_i \in \mathcal{M}(\text{su}(2|2))} v_{g_i}(k) g_i \quad (12)$$

where  $\mathcal{M}(\text{su}(2|2))$  denotes the fermionic sector of  $\text{su}(2|2)$  defined in (10), and if  $g_i$  is one of the first four elements in the set, e.g.  $g_1 = a_+(1-n_-)$ ,

$$\eta v_{g_1}(k) = \sum_{q \in L} \langle k+q, -(k+q) | V | k, -(k+q) \rangle \langle a_{k+q}^\dagger a_{-(k+q)}^\dagger a_{-(k+q)} a_{k+q} \rangle \quad (13)$$

whereas if  $g_i$  is one of the four remaining elements, e.g.  $g_5 = n_+ a_-$ ,

$$\eta v_{g_5}(k) = \sum_{q \in L} \langle k+q, k | V | -k, k \rangle \langle a_{k+q}^\dagger \rangle + \eta v_{g_1}(k) \quad (14)$$

The existence of a dynamical superalgebra permits us to derive in standard fashion both the spectrum of  $H_{\kappa}^{(2)}$  and the self-consistency equations from which the coefficients  $v_{g_i}(k)$  can be determined. Here we report the results for a simplified version of the model, in which it is assumed that  $v_{g_i} = v_1$  and  $v_{i+4} = v_1 + v_2$ ,  $i=1, \dots, 4$ .

The eigenvalues  $\lambda_i$  of  $H$  are the roots of the quartic equation

$$\Delta_{\kappa}^{XY} - Z\{(2\epsilon_{\kappa} - \lambda)X + 2\Delta_{\kappa}(v_1 + v_2)^2\} = 0 \quad (15)$$

where  $X = X(\lambda) = \Delta_{\kappa}(\epsilon_{\kappa} - u_{\kappa} - \lambda)$ ;  $Y = Y(\lambda) = \Delta_{\kappa}(\epsilon_{\kappa} + u_{\kappa} - \lambda)$  and  $Z = Z(\lambda) = 2v_1^2 - \lambda(\epsilon_{\kappa} + u_{\kappa} - \lambda)$ .

The corresponding eigenvectors can be conveniently written as linear combinations of vectors  $\{|n_+, n_-\rangle\}$ , where  $n_+$ ,  $n_-$  are occupation numbers of + and - particles respectively taking values in the set  $\{0,1\}$ , with coefficients in  $\mathbb{R}[\eta]$ :

$$|\lambda_i\rangle = N_i \{ |00\rangle - Z_i/Y_i |11\rangle + \eta \Delta_{\kappa}/X_i Y_i [(v_1 X_i - (v_1 + v_2)Z_i) |10\rangle + (v_1 X_i + (v_1 + v_2)Z_i) |01\rangle] \} ; i=1, \dots, 4 \quad (16)$$

where, for example,  $X_i = X(\lambda_i)$  and  $N_i$  is the normalization factor.

Letting  $v_1, v_2$  go to zero, one recovers the eigenvalues and eigenvectors of  $H_{\kappa}^{(0)}$ .

The self-consistency equations for  $v_1$  and  $v_2$ , using the Gibbs ensemble for  $H_{\kappa}^{(1)}$ , are

$$v_1 \sum_{i=1}^4 \exp(-\lambda_i/k_B T) = \sum_{q \in L} \langle k+q \ - (k+q) | V | k \ - (k+q) \rangle \sum_{i=1}^4 N_i^2 \exp(-\lambda_i/k_B T) \times \\ \times \Delta_{\kappa}^2 / X_i Y_i^2 [-v_1 X_i (Y_i - Z_i) + (v_1 + v_2) Z_i (Y_i + Z_i)] \quad (17)$$

$$v_2 \sum_{i=1}^4 \exp(-\lambda_i/k_B T) = \sum_{q \in L} \langle k+q \ k | V | -k \ k \rangle \sum_{i=1}^4 N_i^2 \exp(-\lambda_i/k_B T) \times \\ \times \Delta_{\kappa}^2 Z_i / X_i Y_i^2 [v_1 X_i + (v_1 + v_2) Z_i] \quad (18)$$

Note that these equations do no more involve the anticommuting variable  $\eta$ . A detailed analysis of the solutions of (17), (18) and their physical meaning will be given elsewhere. We note in passing that the 14 operators (10), (11) generate the 15-dimensional Lie algebra  $su(4)$  under commutation; the additional element generated being the operator  $n_+ n_-$ . However, since the coefficients of the fermionic terms in  $H_{\kappa}^{(2)}$  anticommute with the operators, it would not appear natural to consider  $su(4)$  as the dynamical Lie algebra of this model. In addition the  $su(2) \oplus su(2)$  Lie algebra of the Hamiltonian  $H_{\kappa}^{(1)}$  lies in the even component of the  $su(2|2)$  superalgebra; we now show how  $H_{\kappa}^{(1)}$  can be considered as a supersymmetric Hamiltonian in terms of charge operators  $Q, Q^\dagger$  which lie in  $\mathcal{H}(su(2|2))$ . In order to get the most general element of  $su(2) \oplus su(2)$ , we modify  $H_{\kappa}^{(1)}$  of eq. (6) to

$$\tilde{H}_{\kappa}^{(1)} = \varepsilon_+ n_+ + \varepsilon_- n_- + (\Delta_{\kappa} a_+^\dagger a_-^\dagger + u_{\kappa} a_+^\dagger a_- + \text{h.c.}) \quad (19)$$

by including a  $J'_3$  term  $\varepsilon'_3 (n_+ - n_-)$  (with  $\varepsilon_{\pm} = \varepsilon_{\kappa} \pm \varepsilon'_3$ ; physically this would correspond to the inclusion of an external magnetic field). Now define the charge  $Q$ ,

$$Q = \alpha a_+ n_- + \beta a_- n_+ + \gamma a_+^\dagger (1 - n_-) + \delta a_-^\dagger (1 - n_+) \quad (20)$$

which is an element of  $\mathcal{H}(\text{su}(2|2))$ , and whose square is zero. Then, provided  $|\alpha|^2 - |\gamma|^2 = \varepsilon_-$ ,  $|\beta|^2 - |\delta|^2 = \varepsilon_+$ ,  $-\beta^* \gamma + \alpha^* \delta = \Delta_\kappa$  and  $\delta^* \gamma - \alpha^* \beta = u_\kappa$ , we may express the Hamiltonian (19), up to an additive constant  $c = |\gamma|^2 + |\delta|^2$ , as

$$\tilde{H}_\kappa^{(1)} = \{Q, Q^\dagger\}, \quad \text{with } [\tilde{H}_\kappa^{(1)}, Q] = 0 \quad (21)$$

In this case the potentials must satisfy  $|u_\kappa|^2 = |\Delta_\kappa|^2 + \varepsilon_+ \varepsilon_-$ , and then we have a spontaneously broken supersymmetric model;  $\tilde{H}_\kappa^{(1)}$  satisfies (21), but  $Q^\dagger |f_0\rangle \neq 0$  and  $Q |b_0\rangle \neq 0$  (unless  $c = |u_\kappa| - \sqrt{\varepsilon_+ \varepsilon_-}$ ), where  $\{|f_0\rangle, |b_0\rangle\}$  is the degenerate ground state of the model. In Fig. 1 we exhibit the spectrum of this system and give the states in the basis  $\{|n_+, n_-\rangle\}$ .

If we choose  $\gamma = \delta = 0$ , so that  $\Delta_\kappa = 0$  and  $u_\kappa = e^{i\varphi_\kappa} \sqrt{\varepsilon_+ \varepsilon_-}$ , then we obtain the following Hamiltonian:

$$H_\kappa^{(0)} = \varepsilon_+ n_+ + \varepsilon_- n_- + u_\kappa a_+^\dagger a_- + u_\kappa^* a_-^\dagger a_+ \quad (22)$$

This Hamiltonian describes a system with unbroken supersymmetry, because it is of the form  $H_\kappa^{(0)} = \{Q_0, Q_0^\dagger\}$ , with  $Q_0 = \sqrt{\varepsilon_-} a_+ n_- - e^{-i\varphi_\kappa} \sqrt{\varepsilon_+} a_- n_+$ ,  $Q_0^2 = 0$ , such that  $[H_\kappa^{(0)}, Q_0] = 0$  and both  $Q_0$  and  $Q_0^\dagger$  annihilate the (degenerate) ground state  $\{1/\sqrt{2}(|10\rangle - |01\rangle), |00\rangle\}$ .

Identifying  $\Delta_\kappa$  in (19) as the superconducting order parameter, the Hamiltonian  $H^{(0)}$  describes a system above the critical temperature for pairing, where  $\Delta_\kappa = 0$  and  $|u_\kappa| \neq 0$ . In this sense the superconducting transition may be considered as a spontaneous breaking of supersymmetry for this model. We note in passing that the spectrum generating algebra of  $H_\kappa^{(0)}$  is the algebra  $u(2)$  generated by  $\{1/2(n_+ + n_- - 1), 1/2(n_+ - n_-), a_+^\dagger a_-, a_-^\dagger a_+\}$ . The appropriate superalgebra for discussion of this model is that obtained by adding to these bosonic elements the fermionic elements  $\{a_+ n_-, a_- n_+, n_- a_+^\dagger, n_+ a_-^\dagger\}$  of  $Q_0$  and  $Q_0^\dagger$ . The resulting superalgebra is  $\text{su}(2|1)$  [7]. In Fig. 2 we give a diagram of the connections between the various Lie algebras and superalgebras of the models discussed here.

Figure Captions

Fig. 1 The energy spectrum of  $\tilde{H}_\kappa^{(0)}$ ;  $c = |\gamma|^2 + |\delta|^2$ ,  $e_\pm = |u_\kappa| \pm \sqrt{\epsilon_+ \epsilon_-}$ .

Fig. 2 Lie algebra-superalgebra chain.

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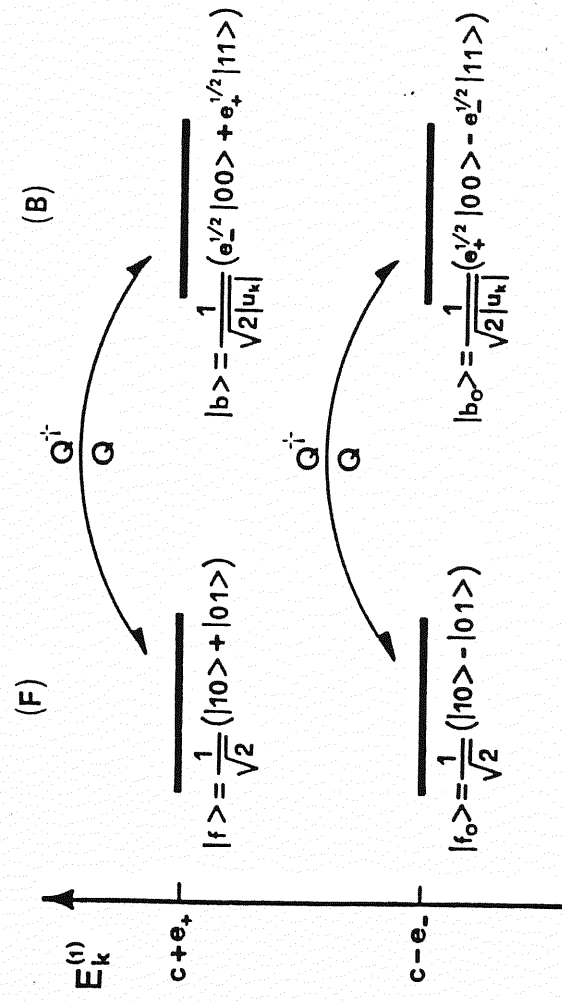


Fig. 1

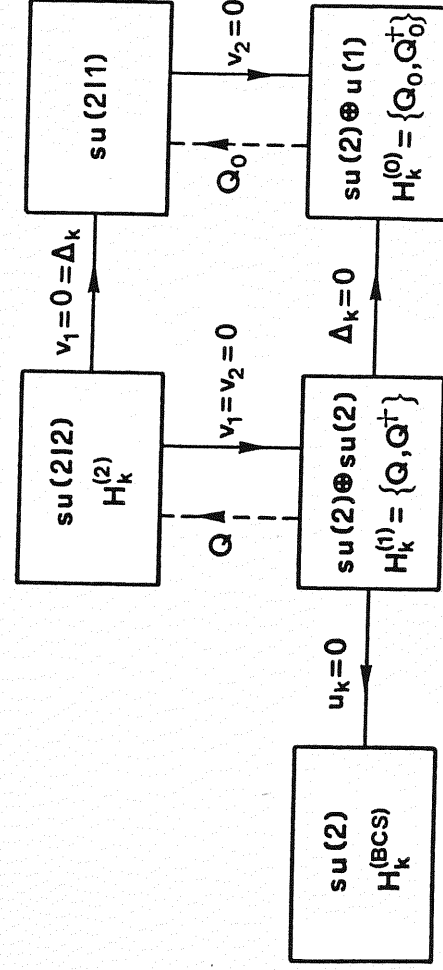


Fig. 2

